

QCD propagators and vertices from lattice QCD

(in memory of Michael Müller-Preußker)

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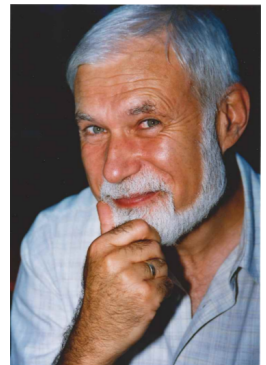
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Abstract. We review lattice calculations of the elementary Greens functions of QCD with a special emphasis on the Landau gauge. These lattice results have been of interest to continuum approaches to QCD over the past 20 years. They are used as reference for Dyson-Schwinger- and functional renormalization group equation calculations as well as for hadronic bound state equations. The lattice provides low-energy data for propagators and three-point vertices in Landau gauge at zero and finite temperature even including dynamical fermions. We summarize Michael Müller-Preußker's important contributions to this field and put them into the perspective of his other research interests.

1 Michael's scientific career and research topics

Michael Müller-Preußker died much too early on October 12, 2015, while on a business trip to a lattice gauge theory workshop in Vladivostok, Russia. Although he had retired four years ago he still held a Senior Professorship at the physics institute of the Humboldt-University Berlin and, active as he was, continued with his research and teaching until the very end. He was a Professor of Theoretical Physics with his heart and soul and a well-appreciated colleague and teacher at his Alma mater.

Michael was born on September 26, 1946 in Potsdam and received the “*Abitur*”, the entrance degree for German universities, in 1965. He enrolled as physics student at Humboldt-University which awarded him the doctoral degree (*Dr. rer. nat.*) in 1973 for his work on “Sum Rules For Helicity Partial Wave Amplitudes” under the supervision of Prof. F. Karsch. He remained at the physics institute also for his postdoctoral studies, employed as a Research Assistant (“*Assistent*” and “*Oberassistent*”) in the research unit *Particles and Fields* from 1972 to 1993. During that time, he was delegated to the Joint Institute for Nuclear Research (JINR) where he spent 5 years as a Visiting Researcher from 1978 to 1983, a period which was very valuable for his scientific career. Ever since then he maintained close relations with his Russian colleagues and there are many joint publications. In 1986 he received the doctoral degree (*Dr. sc. nat.*) for his work on instantons in Euclidean Yang-Mills theory on the lattice and continuum.



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After the German reunification Michael was Visiting Professor at the University of Bielefeld (1990–1991) and then after a short period as Senior Scientist became Professor for Theoretical Physics at the Humboldt-University Berlin in 1993. Since then he educated quite a few physics students and postdocs in his research group on “*Phenomenology / Lattice Gauge Theory*”. In fact, Michael was a passionate and enthusiastic teacher and introduced students to research at an early stage. After his retirement Michael continued with research and teaching on the basis of a Senior Professorship in 2011.

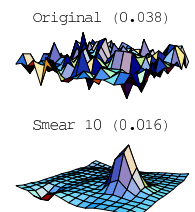
Michael also had quite a few university responsibilities. To name but a few: From 1994 to 1996 he was Vice-President of the Humboldt-University Berlin, a period which was still characterized by dramatic changes in the entire social environment and it did not spare people at the university. From 1997 to 2003 Michael was head of the task force “Adlershof” which organized the relocation of the Institute of Physics from the city center to the new science campus in Berlin-Adlershof. During that time he was also chairman of the Local Organizing Committee of the XIX International Symposium on Lattice Field Theory in 2001. From 2002 to 2006 he was Vice-Dean for Teaching and Studies at the Faculty for Math & Science I and so took part in the introduction of the new Bachelor and Master degrees. Later he helped to reintroduce

the annual graduation ceremony at the institute, a tradition which had disappeared for years. From 2006 to 2010 Michael was Managing Director of the Institute of Physics. After his retirement he was Chairman of the Physical Society Berlin (PGzB) from 2012 to 2014 and member of the scientific board of the North-German Supercomputing Alliance (HLRN).

Michael’s primary research topics were (1) topology in Yang-Mills theories, (2) gauge fixing, in particular the Gribov copy problem, and (3) QCD thermodynamics during the last years. He also looked at other topics like, for example, spin and gauge Higgs models or quasi Monte Carlo methods and many more. Michael always had a good overview about the different developments in the lattice community and from time to time started to work on something new.

Topology was indeed his main subject and his studies were often inspired by semiclassical physics. He became interested in topology during his time in Dubna together with his long-term collaborator Ernst-Michael Ilgenfritz. Their first paper appeared in 1979 and was about the phase transition in the Yang-Mills instanton gas [1]. Topology was also their motivation to start with lattice simulations. In the early 80’s these simulations were just about to gain momentum and their first lattice-based topology results appeared in 1982. First numerical evidence for instantons in the vacuum of a $SU(2)$ lattice gauge theory was then provided in 1986 [2], which was an important contribution at that time.

Many more studies followed, not only on instantons but also on monopoles (e.g., [3–5]) to address the confinement problem in the context of the maximal Abelian gauge and Abelian dominance (dual superconductor scenario by ’t Hooft and Mandelstam), and later also on calorons and dyons. Often these were collaborative work together with Ilgenfritz and their Russian colleagues and long-term collaborators Mitrjushkin, Martemyanov and Bornyakov, or later with van Baal, Bruckmann and Gattringer (e.g., [6, 7]). Michael’s last topology paper was about dyons near the QCD transition temperature [8] and appeared at the end of 2015.



Figures taken from [7]

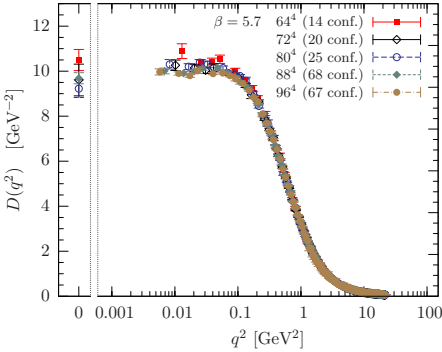


Figure 1. Gluon propagator vs. q^2 ; from [9].

in the gauge condition which was addressed then in many of his subsequent studies. On the lattice this problem is similarly present in the Abelian projection and so as well as relevant for their lattice studies on the dual superconductor scenario. In [5], Gunnar Bali together with Michael and others introduced therefore a new gauge-fixing (maximization) prescription which builds upon the method of simulated annealing to reduce the impact of the Gribov ambiguity. This prescription reappeared later in many of Michael's studies of SU(2) and SU(3) Yang-Mills theories in Landau or Coulomb gauge, for instance, in the lattice studies of the low-momentum behavior of the gluon and ghost propagators which fascinated him for some years. This interest was triggered around 2002 by new developments for the treatment of QCD with functional methods and hence a chance for interdisciplinary research arose. Many lattice studies on that followed and for their efforts the Russian-German group of authors, including Michael together with Bogolubsky, Bornyakov, Ilgenfritz, Mitrushkin and myself, received in a JINR award in 2015 for their successful long-year collaboration. In recent years Michael took part in new investigations of the 3-gluon and quark-gluon vertex in Landau gauge, which is still ongoing.

Since the 90's Michael participated in various lattice calculations in QED or QCD with dynamical fermions. In particular, he was very enthusiastic about QCD thermodynamics. In 2005 this interest became more intensive and focused. A few years before, first large-scale lattice QCD calculations with twisted-mass fermions were performed by members of the nearby NIC/DESY group in Zeuthen to overcome problems of other lattice fermion actions. It was Michael's idea to complement the efforts of the ETM collaboration by corresponding studies at finite temperature, in particular as such studies with Wilson-type fermions were rare at that time.

During the first years, the *tmfT Collaboration*¹ focused on the phase structure in the temperature-mass plane, i.e., the three-dimensional parameter space for temperature, bare quark mass (hopping parameter) and twisted-mass parameter μ [11, 12]. Of special

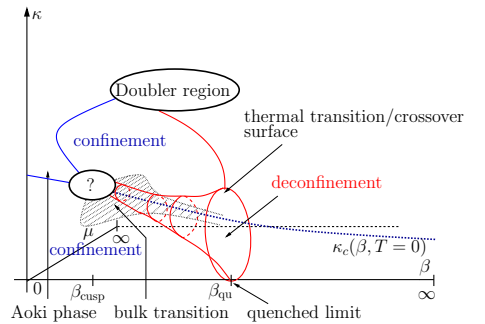


Figure 2. Schematic view of the phase diagram; Figure taken from [11].

¹Members of the *tmfT Collaboration* were: Burger, Ilgenfritz, Jansen, Kirchner, Lombardo, Müller-Preussker, Petschlies, Philipsen, Pinke and Zeidlewicz (though not all at the same time).

interest were the thermal phase transition and the location of the Aoki phase [13].² These preparatory studies turned out to be quite involved due to the richer phase structure in the three-dimensional parameter space (see figure above). Results for the thermal cross-over and for the equation of state appeared then at the end of 2014 with emphasis on the continuum limit because physical quark masses were not yet in reach [16]. This study was then continued with $N_f = 2 + 1 + 1$ fermion flavors to quantify contributions from heavy quarks. In addition to that, the temperature dependence of the topological susceptibility was analyzed. First $N_f = 2 + 1 + 1$ results appeared in a few conference proceedings [17, 18] but this part has remained unfinished since then. An almost up-to-date summary of the tmfT results was given by Florian Burger and Michael in [19] to which I would like to refer the reader for further details.

During the last years Michael also looked at two-color QCD at finite temperature and analyzed with his students (and colleagues) in Berlin and Russia the influence of (1) a magnetic field and (2) that of a finite chiral chemical potential [20]. For those studies dynamical staggered fermions were used instead of Wilson twisted-mass. When switching on a magnetic field they found numerical evidence for magnetic catalysis and inverse catalysis [21, 22] (at different temperatures) which was kind of a hot topic lately.

Michael gave an overview on the present status of his first main topic (topology) at the lattice conference 2014 and devoted this to the memory of Pierre van Baal [23]. In my contribution I will focus on his second main topic and summarize our joint lattice studies (2003–2015). I apologize that I do not have the space to review all developments in this field and concentrate on the lattice studies Michael was involved.

2 Gauge-variant Green's functions of QCD

Lattice-QCD practitioners typically are rather skeptical about the usefulness analyzing gauge-variant quantities on the lattice. Gauge invariance at any finite lattice spacing is the distinguishing feature of the lattice formulation, so why bother worrying about a gauge, related quantities and problems?

If one is just interested in numbers for gauge-invariant quantities like hadron masses, form factors or the like, one indeed does not need to worry about a gauge. An additional condition on the gauge field would simply drop out in the vacuum expectation value of a gauge-invariant object.

The lattice, however, allows for more. It provides nonperturbative access to the elementary n -point Green's functions of QCD whose functional dependence on momentum or distance is intimately connected with a gauge. The quark, gluon and ghost 2-point functions (propagators) in Landau gauge are ever popular examples. Also 3-point functions, like the 3-gluon, quark-gluon or ghost-gluon vertices or any higher n -point vertex function are accessible. The signal-to-noise ratio though decreases the larger n .

Still, why bother about gauge-variant n -point functions? At present I can provide two answers:

1. Lattice calculations have demonstrated great potential for the study of hadronic properties and of QCD in general. Beside the lattice there are however also other nonperturbative frameworks which allow to address strong interaction physics. One example are the flow equation of the Functional Renormalization Group (FRG, aka Wetterich equation [24]), another the bound-state and Dyson-Schwinger equations (DSEs) of QCD.³ As the lattice, both these approaches start

²The Aoki phase for plain Wilson fermions at zero and finite temperature was also the subject of my diploma thesis in 2002 [14, 15]. Michael's supervision and support was excellent and I am very grateful that he introduced me to lattice field theories.

³The reader may consult the reviews [25–28] and the references therein.

from first principles, namely the effective average action and the partition function, respectively, but in addition they require a gauge. Any physics extracted with either method is independent of the gauge, for example, masses or form factors via correspondingly defined bound-state equations. In practice though, the numerical treatment requires a truncation, either of the infinite tower of DSEs or of the infinite hierarchy of integro-differential (FRG) equations. Hence a truncation (and so gauge) dependence may enter.

However, with respect to the lattice these methods also offer some advantages: there are no discretization nor volume effects, changing the quark masses is simple, there is no sign problem targeting the QCD phase diagram and also calculations are not a priori restricted to Euclidean space. They would thus allow us for a look at QCD that complements the lattice, if the truncations were under control. It is hard to quantify their impact or to systematically improve them without external input. But this is where the lattice can help since it allows us to provide that input nonperturbatively and untruncated.

2. Besides this “*lattice service work*” for other scientific communities, calculating these n -point functions also provides access to the strong coupling and quark masses, i.e., the fundamental parameters of QCD. These can be obtained, for example, from the high-momentum dependence of quark, gluon and ghost propagators (see, e.g., [29–33]) or higher n -point functions. Also the chiral condensate $\langle\bar{\psi}\psi\rangle$ can be determined in the massless limit (e.g., [31, 34]). Furthermore, gauge-variant n -point functions are essential for the nonperturbative renormalization of hadron physics observables (e.g., [35]) using RI-MOM schemes [36].

Since n -point functions are the fundamental building blocks of a quantum field theory, for QCD they should show signatures of confinement and chiral symmetry breaking. The chiral condensate is one example, another is the enhancement of the quark mass function one typically sees for the quark propagator in Landau and Coulomb gauges towards zero momentum, i.e., how at low energies a quark becomes a constituent quark. Also confinement should be reflected, and different confinement scenarios connect confinement with a particular behavior of n -point functions. In the 70’s, for example, a $1/p^4$ dependence was expected for the gluon propagator to account for a linear rising quark–anti-quark potential. In contrast, other confinement criteria, namely the Kugo-Ojima scenario and the Gribov-Zwanziger horizon condition, favor an infrared-vanishing gluon propagator and an infrared-diverging ghost dressing function. (We will see below, none of these expectations have been confirmed in lattice simulations, both these objects were found being finite at $p^2 = 0$). So the study of gauge-variant n -point functions gives insight in how confinement and chiral symmetry breaking work in QCD. Sure these functions depend on the gauge but confinement and chiral symmetry should not. There is however no contradiction here, because only the descriptions differ. There is a nice analogy from mechanics: The path through space of a particle in a external field may look different in different coordinate systems, even though they describe the same path.

3 Infrared behavior of the gluon and ghost propagators in Landau gauge

Michael and I got interested in lattice studies of gauge-variant n -point functions around 2003 when Ernst-Michael Ilgenfritz brought the idea from his visit in Tübingen to study low-momentum behavior of the SU(3) gluon and ghost propagators. In Tübingen people were performing similar calculations of these propagators for the SU(2) theory (later in collaboration with Cucchieri and Mendes) [37, 38].

A motivation for this came from new developments for the treatment of the coupled gluon and ghost DSEs by von Smekal, Alkofer and collaborators [26, 39]. Contrarily to former expectations,

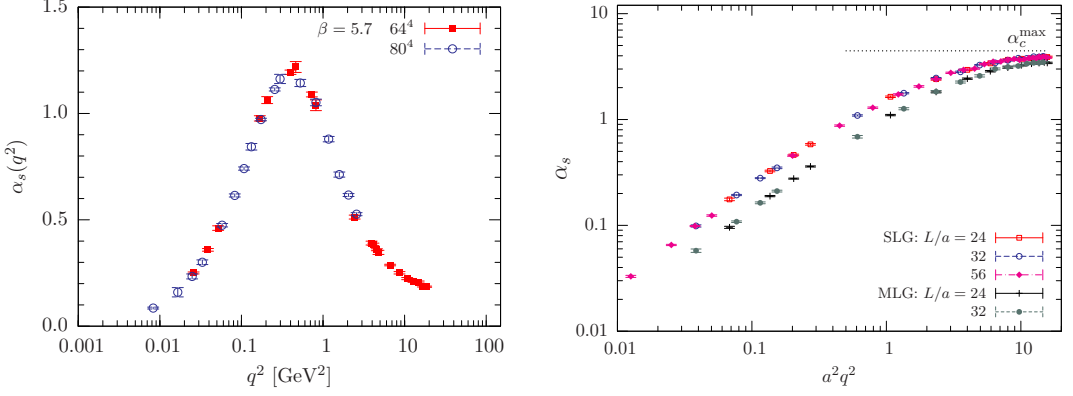


Figure 3. Left from [9]: α_s as defined in Eq. (3) versus q^2 ; Right from [44]: α_s in the strong-coupling limit. If there was a power-law behavior the data points would collapse at $\alpha_s \simeq \alpha_c^{\max}$ for all lattice momenta $a^2 q^2$. The data points (SLG vs. MLG) refer to different discretizations of the gluon and ghost fields.

they found that the gluon propagator does not diverge in the infrared-momentum limit but the ghost propagator should instead diverges stronger than $1/p^2$. In fact, a power-law behavior for the gluon and ghost dressing functions,

$$Z(p^2) \propto (p^2)^{2\kappa} \quad \text{and} \quad J(p^2) \propto (p^2)^{-\kappa} \quad \text{for } p \rightarrow 0 \quad (1)$$

was proposed with a critical exponent $\kappa \simeq 0.595$ [40, 41]. These dressing functions parametrize the nonperturbative momentum dependence of gluon and ghost propagators in Landau gauge

$$D_{\mu\nu}(p^2) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{Z(p^2)}{p^2}, \quad G(p^2) = \frac{J(p^2)}{p^2} \quad (2)$$

and if this power-law behavior (for $p \rightarrow 0$) was true it would nicely fit to the Kugo-Ojima confinement criterion and the Gribov-Zwanziger horizon condition. Furthermore, the strong coupling constant (today aka "Minimal MOM" or "ghost-gluon coupling" [29, 30]) should settle at a finite value

$$\alpha_s(p^2) = \frac{g_0^2}{4\pi} Z(p^2) J^2(p^2) \xrightarrow{p^2 \rightarrow 0} \alpha_c > 0. \quad (3)$$

The first SU(2) lattice results [37, 38] were not in contradiction to this. With our SU(3) lattice calculations we could however reach smaller momenta and clearly saw this coupling to decrease [42], in disagreement to the anticipated power-law behavior. A similar trend was found before by Furui and Nakajima [43], but the error bars of their data still allowed to interpret this as a statistical artifact.

At that time it was not clear if this disagreement at low p^2 simply results from a restricted lattice size. Therefore, calculations for large lattice sizes [up to 128^4 for SU(2) and 96^4 for SU(3)] were launched, in particular of the numerically less demanding gluon propagator. These efforts culminated in 2007 at the Lattice Conference in Regensburg where the disagreement between lattice and functional methods was confirmed: both the SU(2) and SU(3) gluon propagator were found to approach a finite value at zero momentum [45–47] (see also Fig. 1). The ghost propagator was less diverging than expected and the strong coupling constant found to decrease towards $p^2 \rightarrow 0$ (see also Fig. 3).

Besides those efforts on the lattice, different groups also worked on improvements on the functional method side. Some groups even did not see any disagreement, because their solution of the gluon and ghost DSEs qualitatively agrees with the lattice results at small p^2 (see, e.g., [48–50]). Another study of both DSE and FRG equations even found there is a whole family of possible forms for the gluon and ghost propagator’s low-momentum behavior, which can be fixed by an additional condition on the ghost dressing function $J(0)$ at zero momentum [51]. In their language the aforementioned power-law (or conformal) behavior is the *scaling solution* (which one gets when setting $J^{-1}(0) = 0$), while for setting $J^{-1}(0)$ to any finite value one gets one of many *decoupling solutions*, i.e., a gluon propagator and ghost dressing function which are finite at $p^2 = 0$ and a coupling which decreases (as seen on the lattice).

This additional condition on the ghost dressing function seems to cure the Gribov ambiguity in the Landau gauge condition; it definitely triggered new lattice investigations. Studies of the impact of the Gribov ambiguity were not new (Michael and others had performed several before), but now there was a definite Gribov-copy dependence that should be seen for the gluon and ghost propagators at low p^2 [51]. One study was performed by Maas [52], in which the condition on $J(0)$ was mimicked by selecting Gribov copies which at some finite p^2 give large or small values for $J(p^2)$.⁴ Another lattice study was performed by Michael and myself [53], in which we selected Gribov copies based on the lowest-lying (non-zero) eigenvalue λ_1 of the Faddeev-Popov operator.

To better understand how certain copies are selected, it is instructive to remember how the Landau gauge is implemented on the lattice. It is usually a two-step procedure, where in a first step gauge configurations are generated by some standard Monte-Carlo algorithm. In a second step, these (unfixed) configurations of gauge links $\{U_{x\mu}\}$ are then iteratively gauge-transformed

$$U_{\mu}^g(x) = g(x)U_{\mu}(x)g^{\dagger}(x+\mu) \quad (4)$$

along the gauge orbit of $U_{\mu}(x)$ until they satisfy the lattice Landau gauge condition

$$\begin{aligned} 0 &= \sum_{\mu} A_{\mu}^a(x+\mu/2) - A_{\mu}^a(x-\mu/2) \\ &\equiv \sum_{\pm\mu} \Im \text{Tr } T^a U_{\mu}^g(x) \end{aligned} \quad (5)$$

to machine precision. This can be achieved, for instance, by a numerical maximization of the lattice Landau gauge functional (U is kept fixed, g is varied)

$$F_U[g] = \frac{1}{4V} \sum_{x,\mu} \Re \text{Tr } U_{\mu}^g(x). \quad (6)$$

Also saddle points of $F_U[g]$ would satisfy the gauge condition. These however one does not find this way. A maximization, or alternatively an (accelerated) steepest-descent algorithm, will only find one of the many maxima of $F_U[g]$. These constitute a subset of the numerous Gribov copies $U_{\mu}^g(x)$, $U_{\mu}^{g'}(x)$, $U_{\mu}^{g''}(x), \dots$, which all satisfy the Landau gauge condition but are related by a finite gauge transformation. Choosing copies which are global maxima of $F_U[g]$ would for example cure the

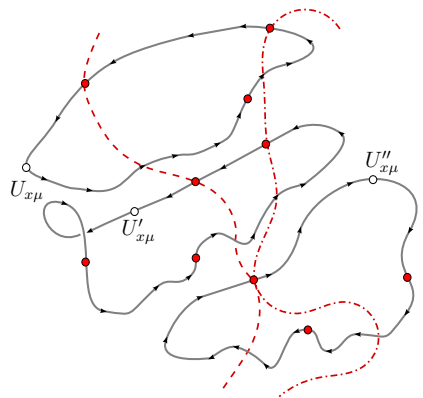


Figure 4. Artist’s rendering of three gauge orbits with depiction of Gribov copies (red circles). A dashed and a dashed-dotted line represent a particular selection of copies, one from each orbit. Such selection is either random or according to a particular criterion (examples are given in the text).

⁴Note that $J(0)$ is not accessible on a periodic lattice due to the trivial zero modes of the Faddeev-Popov operator.

Gribov problem as argued by Zwanziger [54]. This however is numerically hard to achieve. What one can achieve, and was intensively tested by Michael and collaborators, is the method of simulated annealing which usually finds maxima (*best copies*) which are larger than those one finds using a standard gauge-fixing algorithm. It is unclear though if *best copies* are in any way 'better' than others.

Resuming to the discussion before, in [52] Maas selected for each gauge configuration $\{U_\mu(x)\}$ that copy $\{U_\mu^g(x)\}$ which results in a large (or small) vacuum expectation value for $J(p^2)$, irrespective of the value for $F_U[g]$. We [53] chose those copies with the highest (*hc*) or lowest (*lc*) value for λ_1 among 200 copies $\{U_\mu^g(x)\}$ for each configuration $\{U_\mu(x)\}$ (note, *hc* and *lc* refer to the entries in the legend of Fig. 5). Both approaches are somehow complementary, because Gribov copies with small λ_1 give large $J(p^2)$. It was interesting to see that on the lattice one sees this family of decoupling solutions, albeit (for some yet unknown reason) only a small subset and also no scaling behavior is seen. In our study we went even further and did a one-to-one comparison of our lattice data with the (decoupling) DSE results of [51]. We did this comparison separately for two data sets (*hc* and *lc*) and found for both an approximate matching of lattice and DSE results (see Fig. 5).⁵

4 The gluon and ghost propagators at finite temperature

Studying the low-momentum behavior of the gluon and ghost propagators, to some extent, tries to answer an academic question, because for actual hadron physics or QCD thermodynamics calculations with either of the two functional methods, the exact form of low-momentum behavior is not that important. The interesting nonperturbative region is in the mid-momentum regime, i.e., at about 0.5...3 GeV, roughly there where the gluon dressing function shows its characteristic hump (Landau gauge) and where also other gauge-variant n -point functions behave nonperturbatively.

In recent years there have been attempts to address the QCD phase diagram with functional methods (DSEs and FRGEs, see, e.g., [55, 58–60]). As mentioned above, these methods do not suffer from the sign problem as lattice QCD and they can thus address QCD at finite temperature and finite chemical potential more easily. However, these methods have to deal with the fact that the system of equations has to be truncated to perform calculations. These truncations have been improved over the years but without external input it is hard to quantify the effect on the final result.

Together with a PhD student (Aouane) of Michael we have studied the gluon and ghost propagator in Landau gauge at finite temperature. One study was for quenched QCD another for QCD with $N_f = 2$ degenerate (twisted-mass) Wilson fermions. This latter project built upon the vast set of gauge field configurations from the tmfT collaboration. Our results were published in 2011 and 2012 [56, 57] and immediately served as input and benchmark for DSE/FRGE-based studies of the QCD phase diagram (e.g., [55, 59]).

An example is shown in Fig. 6 which has been taken from the DSE-based study of the QCD phase diagram in [55]. In this figure, the DSE results for the transversal and longitudinal gluon dressing functions⁶ are shown for quenched QCD (solid lines) and for QCD with $N_f = 2$ fermions (dashed lines). The DSE solutions for $N_f = 0$ are tuned such that they agree with quenched lattice data (not shown), for example, from [56, 61]. It is reassuring that then their DSE solutions for $N_f = 2$ —using the quenched DSE solutions and a suitable ansatz (truncation) for the fermionic back reaction—more or less match with our $N_f = 2$ lattice data (open symbols in Fig. 6). Our lattice data thus gave strong

⁵There is an interesting story to that: We first thought our lattice results contradict the DSE results, since our gluon propagator data at small p^2 moves up if the ghost dressing function gets a stronger momentum dependence for $p^2 \rightarrow 0$. According to [51] the gluon propagator should decrease instead. It turned out, however, below a certain value for $J(0)$ the DSE results show the same (opposite) trend and also approximately match our data, but those solutions were simply not shown in [51].

⁶Note at finite temperature the Landau-gauge gluon propagator is parametrized by two dressing functions, Z^T and Z^L .

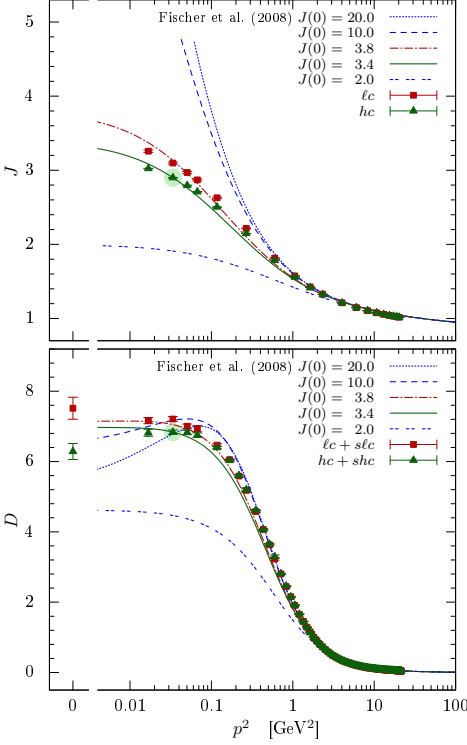


Figure 5. From [53]: The SU(2) ghost dressing function (top) and the gluon propagator (bottom) versus momentum squared. Shown are our lattice data for two different selections of Gribov copies, lines refer to the DSE solutions of [51].

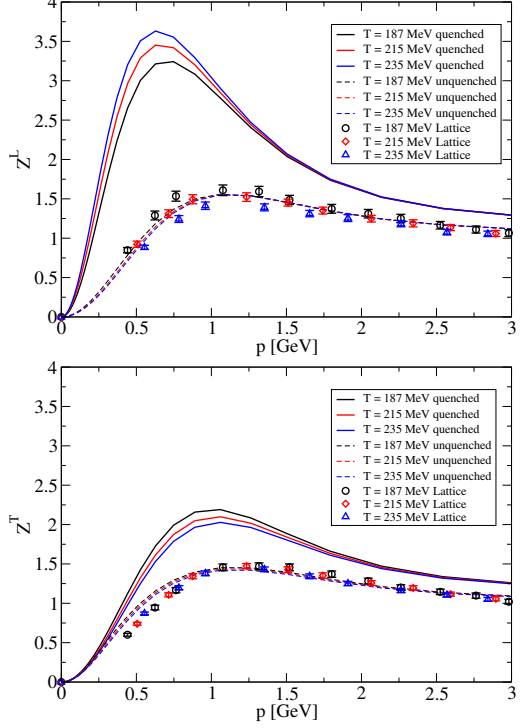


Figure 6. From [55]: Longitudinal (Z^L , top) and transversal (Z^T , bottom) gluon dressing function vs. p . Solid lines are DSE solutions tuned such that they match our quenched lattice results (not shown) from [56]. Dashed lines are DSE solutions which incorporate quark effects. Open symbols are our $N_f = 2$ data from [57].

support for the truncation used in [55], based on which an approximate location of the critical end point in the QCD phase diagram could be provided.

5 Triple-gluon and quark-gluon vertex in Landau gauge

For similar efforts in the future it is important to proceed with lattice calculations of gauge-variant n -point functions. Basically for all n -point functions with $n > 2$ not much is known about the nonperturbative structure and lattice input is much appreciated. The 3-gluon and quark-gluon vertex at zero and finite temperature, for example, are good next candidates. They are of prime importance for the study of bound-state equations, for studies of QCD at finite temperature and chemical potential and implicitly are always also needed for the developments of improved truncation schemes (see, e.g., [28] for truncations beyond 'Rainbow-Ladder').

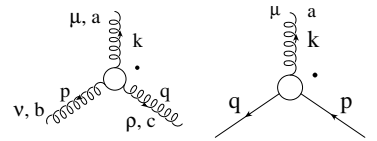


Figure 7. Triple-gluon and quark-gluon vertex; taken from [26].

Compared to 2-point functions (propagators), lattice calculations of vertex functions are much more involved, however. To estimate the momentum dependence of, say the gluon propagator, one needs about 100 well-decorrelated gauge field configurations (if translation invariance is exploited). Moreover, a propagator is typically parametrized by only one or two form factors, and these one often gets straightforward from the Monte-Carlo estimate. A vertex, in comparison, comes with a much richer tensor structure and its form factors in general depend on at least two momenta. Furthermore, to reliably estimate the corresponding n -point Green's function a much increased statistics is needed (for our results in Fig. 8 we had to analyze about 1000 gauge field configurations for each parameter set). Another complication is the fact that it is the n -point (Green's) function one calculates on the lattice, not the vertex. The latter is extracted from certain combinations of the Monte-Carlo estimates for the 2- and 3-point functions. In fact, lattice studies give us access to the quark propagator (S), the gluon propagator ($D_{\mu\nu}$) and to the 3-point Green's functions

$$G_{\mu\nu\rho}(p, q) = \sum_{xy} e^{-ipx-iqy} \langle A_\mu(x) A_\nu(y) A_\rho(0) \rangle = D_{\mu\lambda}(p) D_{\nu\sigma}(q) D_{\rho\omega}(-p-q) \Gamma_{\lambda\sigma\omega}^{AAA}(p, q), \quad (7)$$

$$V_\mu(p, q) = \sum_{xy} e^{-ipx-iqy} \langle \bar{\psi}(x) \psi(y) A_\mu(0) \rangle = S(p) S(q) D_{\mu\nu}(-p-q) \Gamma_\nu^{\bar{\psi}\psi A}(p, q). \quad (8)$$

The last terms demonstrate, the vertex Γ is obtained through an amputation of the corresponding propagators. It is this amputated (and often statistically noisy) object from which one extracts the several form factors of the vertex.

The 3-gluon and quark-gluon vertex are parametrized by 14 and 12 form factors, respectively,⁷

$$\Gamma_{\mu\nu\lambda}^{AAA}(p, q) = \sum_{i=1}^{14} f_i(p, q) P_{\mu\nu\lambda}^{(i)}(p, q), \quad (9)$$

$$\Gamma_\mu^{\bar{\psi}\psi A}(p, q) = \sum_{i=1}^4 \lambda_i(p, q) L_\mu^{(i)}(p, q) + \sum_{j=1}^8 \tau_j(p, q) T_\mu^{(j)}(p, q). \quad (10)$$

The P s, L s and T s are suitably chosen base vectors in the momenta p and q (see, e.g., [62, 63]) and the scalars f_i , λ_i and τ_i are the corresponding form factors which contain all the nonperturbative information of the vertex. They are functions of p^2 , q^2 and $p \cdot q$ and, depending on the base and momenta p and q , are accessible through Eqs. (7) and (8). From perturbation theory, the (off-shell) form factors are known up two-loop order (see, e.g., [63] are references therein). For special choices of momenta even three-loop results are available [64].

In Landau gauge, only the transverse part of the 3-gluon vertex is relevant, because it is always sandwiched by transverse gluons. The transversally projected 3-gluon vertex can be parametrized by four (transverse) form factors $F_{1,\dots,4}$ (see, e.g., [65]). These form factors are accessible on the lattice, in the past, however, only the projection

$$G_1(p, q) = \frac{\Gamma_{\mu\nu\rho}^{(0)}}{\Gamma_{\mu\nu\rho}^{(0)} D_{\mu\lambda}(p) D_{\nu\sigma}(q) D_{\rho\omega}(p-q) \Gamma_{\lambda\sigma\omega}^{(0)}} G_{\mu\nu\rho}(p, q, p-q) \quad (11)$$

of the vertex onto its tree-level form, $\Gamma^{(0)}(p, q)$, has been considered. For SU(2) Yang-Mills theory it was found [66] that G_1 shows a zero-crossing at some small momentum $|p| = |q|$. This was unexpected

⁷For $\Gamma_\mu^{\bar{\psi}\psi A}$ we quoted the "Ball-Chiu" form, according to which the vertex splits into a transverse part (second term) and another that satisfies the Slavnov-Taylor identity (first term).

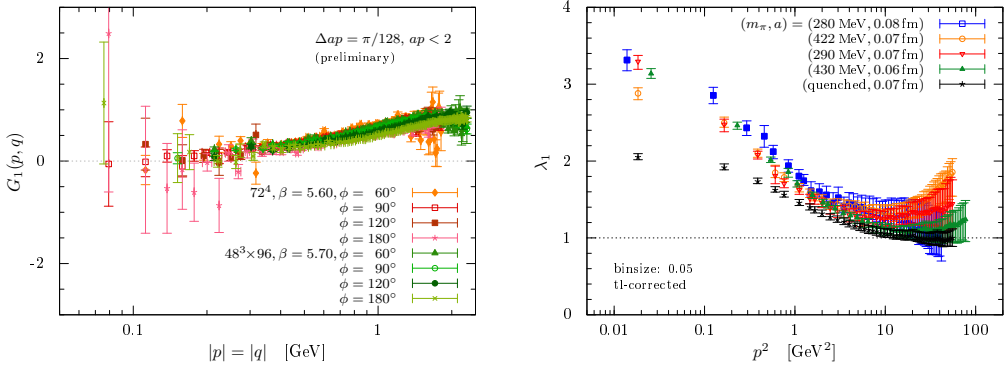


Figure 8. Form factors versus momentum, which describe the deviation from the tree-level form of the 3-gluon vertex (left) and of the quark-gluon vertex (right). The data is yet preliminary and not renormalized. For the 3-gluon vertex we show data for different angles ϕ between p and q . For the quark-gluon-vertex only the soft-gluon kinematic (gluon has zero momentum) is shown.

and triggered some interest. Recently, this zero-crossing has been confirmed by two independent lattice studies [67, 68].

In 2014 Michael and I were granted computing time for a study of the 3-gluon and quark-gluon vertex. Our plan has been to study not only a particular channel but the full nonperturbative structure of both vertices. Since then many $N_f = 2$ gauge field configurations (provided by the RQCD collaboration) have been fixed to Landau gauge and the necessary 2- and 3-point functions were calculated. We further generated some quenched ensembles to analyze unquenching effects and to reach much lower momenta. The analysis of the 3-gluon vertex was done in collaboration with a student of mine (Balduf). The study of the quark-gluon vertex is a collaborative effort with colleagues from Australia (Kızılersü, Williams), Portugal (Oliveira, Silva) and Irland (Skullerud).

For both vertices we have first (preliminary) data. For example, for the 3-gluon vertex we see that the leading (transverse) form factor is F_1 while the remaining three are much smaller in comparison and also show only a weak momentum dependence (we studied cases where $|p| = |q|$). That is, $G_1 \approx F_1$ contains most of the 3-gluon vertex's nonperturbative information. Furthermore we have looked at the zero-crossing, but cannot confirm it yet (see Fig. 8, left). Our current data for G_1 approaches zero but does not clearly cross it (taking the numerical fluctuations as such). This remains to be clarified and will be discussed in due course [69].

Also for the quark-gluon vertex preliminary data for the soft-gluon kinematic (zero gluon momentum) is available. In this kinematic only the form factors λ_1 , λ_2 and λ_3 are non-zero. An example of the (preliminary) data for λ_1 is shown in Fig. 8, right; it is the same as in [70]. One sees we have to deal with quark mass effects and significant discretization effects at larger momenta but apart from that, the data at lower momenta clearly shows the quark-gluon vertex becomes quickly nonperturbative if p^2 drops below 2 or 3 GeV^2 . Similar is seen for λ_2 and λ_3 . Above this momentum regime, the quark-gluon vertex seems to share much with its tree-level form. Also this will be discussed in more detail in due course [71].

Let me conclude this brief summary of our lattice studies of gauge-variant n -point functions. I feel grateful for the time spent with Michael and our collaborators as we traveled together on this 14 year endeavor. We will miss a passionate teacher, a distinguished scientist, a person of integrity and a faithful colleague.

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